New connection formulae for the *q*-orthogonal polynomials via a series expansion of the *q*-exponential

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Abstract
Using a realization of the *q*-exponential function as an infinite multiplicative series of the ordinary exponential functions we obtain new *nonlinear* connection formulae of the *q*-orthogonal polynomials such as *q*-Hermite, *q*-Laguerre and *q*-Gegenbauer polynomials in terms of their respective classical analogues.

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1. Introduction

With the advent of quantum groups the *q*-orthogonal polynomials are objects of special interest in both mathematics and physics. For instance, a *q*-deformed harmonic oscillator provides [1] a group-theoretic setting for the *q*-Hermite and the *q*-Laguerre polynomials. Moreover, the *q*-Hermite polynomials naturally arise [2] in the context of the eigenfunctions of the Hamiltonian of a *q*-deformed harmonic oscillator. Similarly in the theory of the *q*-analogue of the hydrogen atom [3] the *q*-Laguerre polynomials are expected to play an essential role. Another example is provided by the *q*-Gegenbauer polynomials which are the building blocks of the correlation functions of various deformed integrable models [4, 5]. In one way the *q*-deformed systems may be physically viewed as perturbations of the corresponding undeformed ($q = 1$) systems leading to interesting properties of the spectrum [3]. Also in the context of the semiclassical (WKB) approximation it has been observed [6] that for a real *q* the WKB equivalent potential of a *q*-deformed oscillator at slightly off-equilibrium position resembles a classical sextic oscillator. In this scenario suitable nonlinear connection formulae interrelating the *q*-orthogonal polynomials with their classical partners may be useful in extracting the classical perturbative terms reproducing similar spectrum as that of the *q*-deformed systems. Furthermore such connection formulae may establish direct linkages between the correlation functions of the *q*-deformed integrable systems and their classical analogues leading to an understanding of the deformations that preserve integrability. Influenced by these observations here we provide a general framework for obtaining these

nonlinear connection formulae, and, in particular, explicitly derive them for q-Hermite, q-Laguerre and q-Gegenbauer polynomials in terms of their respective classical counterparts.

In the mathematical framework needed to describe the properties of these q-polynomials, such as the recurrence relations, generating functions and orthogonality relations, Jackson's q-exponential plays a key role. Recently, Quesne [7] has expressed Jackson's q-exponential as a multiplicative series of the ordinary exponentials with known coefficients in closed form. This result is the tool of our choice for investigating the nonlinear connection formulae mentioned earlier.

Jackson actually introduced two related q-exponentials:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n, \qquad E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n, \tag{1.1}$$

where

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k). \tag{1.2}$$

The classical limits of the above q-exponentials read

$$\lim_{q \longrightarrow 1} e_q((1-q)z) = \exp(z), \qquad \lim_{q \longrightarrow 1} E_q((1-q)z) = \exp(z). \tag{1.3}$$

Heine's q-binomial theorem provides [8] the following multiplicative series:

$$e_q(z) = \frac{1}{(z;q)_\infty}, \qquad E_q(z) = (-z;q)_\infty, \tag{1.4}$$

such that

$$e_q(z)E_q(-z) = 1. \tag{1.5}$$

In physics literature another form of q-exponential commonly occurs. This is given by

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} z^n, \tag{1.6}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q}, \qquad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \qquad [0]_q! = 1. \tag{1.7}$$

Comparing the first equation in (1.1) and (1.6) it is evident that

$$\exp_q(z) = e_q((1-q)z). \tag{1.8}$$

Quesne [7] expressed the q-exponential (1.6) as a product series of the ordinary exponentials as follows:

$$\exp_q(z) = \exp\left(\sum_{k \in \mathbb{N}} c_k(q)z^k\right), \qquad c_k(q) = \frac{(1-q)^{k-1}}{k[k]_q}. \tag{1.9}$$

This expansion allows us to write

$$e_q(z) = \exp\left(\sum_{k \in \mathbb{N}} \frac{z^k}{k(1 - q^k)}\right), \tag{1.10}$$

and

$$E_q(z) = \exp\left(\sum_{k \in \mathbb{N}} \frac{(-1)^{k+1}z^k}{k(1 - q^k)}\right). \tag{1.11}$$

These results are particularly suitable for applications in the theory of special functions. Specifically, they may be readily employed to develop a general procedure for obtaining connection formulae for various q-orthogonal polynomials. First, the generating function of a suitable q-orthogonal polynomial may be expressed as an infinite product of generating functions of the corresponding classical polynomial. This, in turn, may be utilized to develop a Diaphontine partition equation. As successive terms in the said product series have increasing exponents of the arguments, the partition equation has a finite number of solutions and each q-orthogonal polynomial can be expressed in terms of its classical counterparts of equal or lesser orders. In the following sections we illustrate this procedure in the cases of q-Hermite, q-Laguerre and q-Gegenbauer polynomials.

2. q-Hermite polynomials

It has been observed [2] that the q-Hermite polynomials are components of the eigenfunctions of the Hamiltonian of the q-deformed harmonic oscillator. These authors investigated the recurrence relation, the generating function, the orthogonality relation and other properties of the said polynomials. The generating function of the q-Hermite polynomials has been expressed [2] in terms of the q-exponentials:

$$\mathsf{G}_q(z;t) \equiv E_{q^{-2}}(2(1-q^{-2})zt)e_{q^{-4}}\left(-\frac{2(1-q^{-4})t^2}{q(1+q^{-2})}\right)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n/2}}{[n]_{q^{-2}}!} H_n(z;q)t^n. \tag{2.1}$$

In the $q \to 1$ limit, above function $\mathsf{G}_q(z;t)$ yields the well-known generating function of the classical Hermite polynomials:

$$\mathsf{G}(z;t) \equiv \exp(2zt - t^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(z)t^n. \tag{2.2}$$

For later use in our expansion scheme the classical Hermite polynomials are listed below:

$$H_n(z) = \sum_{\ell=0}^{[n/2]} (-1)^\ell \frac{n!(2z)^{n-2\ell}}{\ell!(n-2\ell)!}, \tag{2.3}$$

where the symbol $[n/2]$ means the largest integer smaller or equal to $n/2$. Following (1.10) and (1.11) we recast the q-exponentials in (2.1) as

$$E_{q^{-2}}(2(1-q^{-2})zt) = \exp\left(\sum_{k\in\mathbb{N}} (-1)^{k+1} c_k(q^{-2})(2zt)^k\right),$$

$$e_{q^{-4}}\left(-\frac{2(1-q^{-4})t^2}{q(1+q^{-2})}\right) = \exp\left(\sum_{k\in\mathbb{N}} (-1)^k c_k(q^{-4})\left(\frac{2t^2}{q(1+q^{-2})}\right)^k\right). \tag{2.4}$$

Introducing the following parameters

$$\tau_k = (-1)^{(k+1)/2}\left(\frac{2}{q[2]_{q^{-2}}}\right)^{k/2}\sqrt{c_k(q^{-4})}\, t^k,$$

$$\zeta_k = (-1)^{(k+1)/2}(2q[2]_{q^{-2}})^{k/2}\frac{c_k(q^{-2})}{2\sqrt{c_k(q^{-4})}} z^k, \tag{2.5}$$

Table 1. Contributions to $H_5(z; q)$.

Solutions of (2.8) for $n = 5$	Contributions to $H_5(z; q)$
$n_1 = 5$	$-\frac{[5]_{q^{-2}}!}{5!}\left(\frac{2}{q^2[2]_{q^{-2}}}\right)^{5/2} H_5(\zeta_1)$
$n_1 = 3, n_2 = 1$	$\sqrt{-1}\frac{[5]_{q^{-2}}!}{3!}\left(\frac{2}{q^2[2]_{q^{-2}}}\right)^{5/2}\sqrt{c_2(q^{-4})}H_3(\zeta_1)H_1(\zeta_2)$
$n_1 = 2, n_3 = 1$	$\frac{[5]_{q^{-2}}!}{2!}\left(\frac{2}{q^2[2]_{q^{-2}}}\right)^{5/2}\sqrt{c_3(q^{-4})}H_2(\zeta_1)H_1(\zeta_3)$
$n_1 = 1, n_4 = 1$	$-\sqrt{-1}[5]_{q^{-2}}!\left(\frac{2}{q^2[2]_{q^{-2}}}\right)^{5/2}\sqrt{c_4(q^{-4})}H_1(\zeta_1)H_1(\zeta_4)$
$n_2 = 1, n_3 = 1$	$-\sqrt{-1}[5]_{q^{-2}}!\left(\frac{2}{q^2[2]_{q^{-2}}}\right)^{5/2}\sqrt{c_2(q^{-4})c_3(q^{-4})}H_1(\zeta_2)H_1(\zeta_3)$
$n_1 = 1, n_2 = 2$	$\frac{[5]_{q^{-2}}!}{2!}\left(\frac{2}{q^2[2]_{q^{-2}}}\right)^{5/2}c_2(q^{-4})H_1(\zeta_1)H_2(\zeta_2)$
$n_5 = 1$	$-[5]_{q^{-2}}!\left(\frac{2}{q^2[2]_{q^{-2}}}\right)^{5/2}\sqrt{c_5(q^{-4})}H_1(\zeta_5)$

the generating function (2.1) may be expressed as a multiplicative series of the classical generating functions:

$$\mathsf{G}_q(z; t) = \prod_{k\in\mathbb{N}} \mathsf{G}(\zeta_k; \tau_k). \tag{2.6}$$

Employing expansions (2.1) and (2.2) of the above generating functions and comparing coefficients of equal power of t on both sides, we obtain a connection formula for the q-Hermite polynomials in terms of its classical partners of lower dimensions:

$$\frac{q^{n/2}}{[n]_{q^{-2}}!}H_n(z; q) = \left(\frac{2}{q[2]_{q^{-2}}}\right)^{n/2}\sum_{\substack{n_1,n_2,\cdots\\=0}}^{\infty}(-1)^{(n+\sum_{k\in\mathbb{N}}n_k)/2}\prod_{k\in\mathbb{N}}\left[(c_k(q^{-4}))^{n_k/2}\frac{H_{n_k}(\zeta_k)}{n_k!}\right]\delta_{\sum_{k\in\mathbb{N}}kn_k,n}. \tag{2.7}$$

The solutions of Diophantine partition relation

$$\sum_{k\in\mathbb{N}} kn_k = n \tag{2.8}$$

select the product structure of the classical Hermite polynomials appearing on the rhs of (2.7). As an illustration we here use the connection formula (2.7) for reconstructing the q-Hermite polynomial $H_5(z; q)$. The solution of the partition equation (2.8) and the corresponding contributions to the connection formula (2.7) are listed in table 1. Combining the entries of the second column we may easily reproduce the well-known [2] result for $H_5(z; q)$:

$$H_5(z; q) = 32q^{-45/2}z^5 - 16q^{-19/2}[2]_{q^{-4}}[5]_{q^{-2}}z^3 + 8q^{-9/2}[3]_{q^{-2}}[5]_{q^{-2}}z. \tag{2.9}$$

3. q-Laguerre polynomials

The theory of the q-Laguerre polynomials has been studied [1, 9] extensively. They appear [1] in the representation theory of the enveloping algebra of the q-deformed Heisenberg algebra. These authors observed [1] that the generating function of q-Laguerre polynomials may be cast in the form

$$\mathcal{G}_q^{(n)}(z; t) \equiv E_q(-(1-q)zt)\left(-\frac{q}{t}; q\right)_n t^n = \sum_k q^{(n-k)(n-k+1)/2}L_k^{(n-k)}(z; q)t^k, \tag{3.1}$$

which may be viewed as the q-analogue of the generating function [10] of the classical Laguerre polynomials:

$$\mathcal{G}^{(n)}(z;t) \equiv \exp(-zt)(1+t)^n = \sum_k L_k^{(n-k)}(z)t^k. \tag{3.2}$$

For the purpose of later use we list here the classical Laguerre polynomials as

$$L_k^{(n-k)}(z) = \sum_{\ell=0}^{k}(-1)^\ell \binom{n}{k-\ell}\frac{z^\ell}{\ell!}. \tag{3.3}$$

The product structure (1.11) may now be utilized to express the q-Laguerre generating function (3.1) as a multiplicative series of the classical Laguerre generating function (3.2):

$$\mathcal{G}_q^{(n)}(z;t) = \prod_{k\in\mathbb{N}}[\mathcal{G}^{(n_k)}(c_k(q)z^k;t^k)(1+t^k)^{-n_k}]\left(-\frac{q}{t};q\right)_n t^n, \tag{3.4}$$

where any integer set $\{n_k\}$ may be used in the rhs. Each set of auxiliary integer parameters $\{n_k\}$ provides an expansion scheme for the q-Laguerre polynomials. The deformed q-Laguerre polynomials, when reconstructed via our expansion scheme involving the classical Laguerre polynomials, must not depend on the intermediate auxiliary integer parameters $\{n_k\}$. As observed below, precisely this happens. The expansion scheme (3.4) forms the key ingredient of our method. To explicitly obtain the connection formula for the q-Laguerre polynomials we proceed by expressing the q-binomial theorem [8] as

$$\left(-\frac{q}{t};q\right)_n t^n = \sum_{\ell=0}^{n} q^{(n-\ell)(n-\ell+1)/2}\begin{bmatrix}n\\\ell\end{bmatrix}_q t^\ell, \tag{3.5}$$

where $\begin{bmatrix}n\\\ell\end{bmatrix}_q = \frac{[n]_q!}{[\ell]_q![n-\ell]_q!}$. Using Pochhammer symbol

$$(\alpha)_\ell = \prod_{j=0}^{\ell-1}(\alpha+j), \qquad (\alpha)_0 = 1, \tag{3.6}$$

we rewrite

$$(1+t)^{-n} = \sum_{\ell=0}^{\infty}(-1)^\ell \frac{(n)_\ell}{\ell!}t^\ell. \tag{3.7}$$

Systematic use of results (3.2), (3.5) and (3.7) on the rhs of the expansion scheme (3.4) now yields the relation

$$\mathcal{G}_q^{(n)}(z;t) = \sum_{\ell=0}^{n}\sum_{\{k_j\}}\sum_{\{\ell_j\}}\prod_{j\in\mathbb{N}}\left((-1)^{\ell_j}\frac{(n_j)_{\ell_j}}{\ell_j!}L_{k_j}^{(n_j-k_j)}(c_j(q)z^j)t^{j(k_j+\ell_j)}\right)q^{(n-\ell)(n-\ell+1)/2}\begin{bmatrix}n\\\ell\end{bmatrix}_q t^\ell. \tag{3.8}$$

Using the second equality in (3.1) and comparing the coefficients of t^k on both sides of (3.8), we obtain the promised nonlinear connection formula:

$$q^{(n-k)(n-k+1)/2}L_k^{(n-k)}(z;q) = \sum_{\ell=0}^{n}\sum_{\{k_j\}}\sum_{\{\ell_j\}}\prod_{j\in\mathbb{N}}\left((-1)^{\ell_j}\frac{(n_j)_{\ell_j}}{\ell_j!}L_{k_j}^{(n_j-k_j)}(c_j(q)z^j)\right)$$
$$\times q^{(n-\ell)(n-\ell+1)/2}\begin{bmatrix}n\\\ell\end{bmatrix}_q \delta_{\sum_j j(k_j+\ell_j)+\ell,k}. \tag{3.9}$$

Table 2. Contributions to $L_3^{(0)}(z; q)$.

Solutions of (3.10)	Contributions to $L_3^{(0)}(z; q)$
$k_1 = 3$	$q^6 L_3^{(n_1-3)}(z)$
$\ell_1 = 3$	$-q^6 \frac{(n_1)_3}{3!}$
$\ell = 3$	1
$k_3 = 1$	$q^6 L_1^{(n_3-1)}(c_3(q)z^3)$
$\ell_3 = 1$	$-q^6 n_3$
$k_1 = 1, k_2 = 1$	$q^6 L_1^{(n_1-1)}(z) L_1^{(n_2-1)}(c_2(q)z^2)$
$\ell_1 = 1, \ell_2 = 1$	$q^6 n_1 n_2$
$k_1 = 2, \ell_1 = 1$	$-q^6 n_1 L_2^{(n_1-2)}(z)$
$k_1 = 1, \ell_1 = 2$	$q^6 \frac{(n_1)_2}{2} L_1^{(n_1-1)}(z)$
$k_1 = 2, \ell = 1$	$q^3 [3]_q L_2^{(n_1-2)}(z)$
$k_1 = 1, \ell = 2$	$q[3]_q L_1^{(n_1-1)}(z)$
$\ell = 1, \ell_1 = 2$	$q^3 [3]_q \frac{(n_1)_2}{2}$
$\ell = 2, \ell_1 = 1$	$-q[3]_q n_1$
$k_2 = 1, \ell_1 = 1$	$-q^6 n_1 L_1^{(n_2-1)}(c_2(q)z^2)$
$k_2 = 1, \ell = 1$	$q^3 [3]_q L_1^{(n_2-1)}(c_2(q)z^2)$
$\ell = 1, \ell_2 = 1$	$-q^3 [3]_q n_2$
$k_1 = 1, \ell_2 = 1$	$-q^6 n_2 L_1^{(n_1-1)}(z)$
$k_1 = 1, \ell = 1, \ell_1 = 1$	$-q^3 [3]_q n_1 L_1^{(n_1-1)}(z)$

The solutions of the Diophantine partition relation

$$\sum_{j \in \mathbb{N}} j(k_j + \ell_j) + \ell = k \tag{3.10}$$

determine the set of classical Laguerre polynomials contributing to the expansion of a particular q-Laguerre polynomial. In continuation of the discussion following (3.4) we note that the lhs of (3.9) is independent of the set $\{n_j\}$, and, consequently, each set of allowed $\{n_j\}$ provides an expansion of the q-Laguerre polynomial $L_k^{(n-k)}(z; q)$. This is a generic feature of our procedure. Equations (3.9) and (3.10) form the main results of this section. To illustrate our process with an example we here enumerate the contributing terms for $n = 3$, $k = 3$. The corresponding solutions of the partition relation (3.10) and their respective contributions to $L_3^{(0)}(z; q)$ are listed in table 2. The entries on the first column refer to the non-zero elements in the solutions of the partition equation (3.10) for $k = 3$. Using the classical Laguerre polynomials given in (3.3), we may now directly recover the polynomial $L_3^{(0)}(z; q)$ by summing the entries on the second column of table 2. As noted earlier, the dependences on the auxiliary integer parameters n_1, n_2, n_3 in the contributions to $L_3^{(0)}(z; q)$, when summed, disappear. Summing the contributions displayed on the second column in table 2 we obtain

$$L_3^{(0)}(z; q) = 1 - q \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q z + q^4 \frac{1}{[2]_q!} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q z^2 - q^9 \frac{1}{[3]_q!} z^3, \tag{3.11}$$

which, of course, agrees with the well-known [1, 9] result. This validates our expansion scheme of the q-Laguerre polynomials in terms of the classical Laguerre polynomials.

4. q-Gegenbauer polynomials

Classical Gegenbauer polynomials and their generalizations appear in many areas of theoretical physics, such as the correlation functions for the Kniznik–Zamolodchikov (KZ) [11] equation for the $\widehat{sl}_2$ algebra and the wavefunctions of the integrable systems [12] generalized from the Calogero–Sutherland [13, 14] model. Deformed q-Gegenbauer polynomials have been introduced by Askey and Ismail [15]. Naturally, these deformed polynomials are of interest in studying the q-KZ equation for the $U_q(\widehat{sl}_2)$ algebra [4] and also the deformations of integrable models [5]. We think our method of expressing q-Gegenbauer polynomials as nonlinear superpositions of classical Gegenbauer polynomials may be of use in these studies.

The generating function [15] of the q-Gegenbauer polynomials read

$$
\begin{aligned}
\mathfrak{G}_q^{(\lambda)}(\cos\theta, t) &\equiv \frac{(q^\lambda \exp(i\theta)t; q)_\infty}{(\exp(i\theta)t; q)_\infty} \frac{(q^\lambda \exp(-i\theta)t; q)_\infty}{(\exp(-i\theta)t; q)_\infty} \\
&= \frac{E_q(-q^\lambda \exp(i\theta)t)}{E_q(-\exp(i\theta)t)} \frac{E_q(-q^\lambda \exp(-i\theta)t)}{E_q(-\exp(-i\theta)t)} \\
&= \sum_{n=0}^\infty C_n^{(\lambda)}(\cos\theta; q)t^n.
\end{aligned} \tag{4.1}
$$

The q-Gegenbauer polynomials are explicitly given by [15]

$$
C_n^{(\lambda)}(\cos\theta; q) = \sum_{\ell=0}^n \frac{(q^\lambda; q)_\ell (q^\lambda; q)_{n-\ell}}{(q; q)_\ell (q; q)_{n-\ell}} \cos(n - 2\ell)\theta. \tag{4.2}
$$

In contrast to the case of Hermite and Laguerre polynomials discussed earlier the generating function for the classical Gegenbauer polynomials is not usually expressed in terms of ordinary exponentials:

$$
\mathfrak{G}^{(\lambda)}(\cos\theta, t) \equiv (1 - 2\cos\theta t + t^2)^{-\lambda} = \sum_{n=0}^\infty C_n^{(\lambda)}(\cos\theta)t^n. \tag{4.3}
$$

For subsequent applications we here list the classical Gegenbauer polynomials as follows:

$$
C_n^{(\lambda)}(\cos\theta) = \sum_{\ell=0}^n \frac{(\lambda)_\ell (\lambda)_{n-\ell}}{\ell!(n-\ell)!} \cos(n - 2\ell)\theta. \tag{4.4}
$$

To establish a connection formula between the q-deformed Gegenbauer polynomials and their classical analogues we recast the classical generating function (4.3) in an alternate form:

$$
\mathfrak{G}^{(\lambda)}(\cos\theta, t) = \exp\left(2\lambda \sum_{k\in\mathbb{N}} \cos(k\theta) \frac{t^k}{k}\right). \tag{4.5}
$$

In a parallel construction we use the expansion scheme (1.11) to express the deformed generating function $\mathfrak{G}_q^{(\lambda)}(\cos\theta, t)$ given in (4.1) as an infinite product series of the ordinary exponentials:

$$
\mathfrak{G}_q^{(\lambda)}(\cos\theta, t) = \exp\left(2 \sum_{k\in\mathbb{N}} [\lambda]_{q^k} \cos(k\theta) \frac{t^k}{k}\right). \tag{4.6}
$$

The close kinship between the product serieses (4.5) and (4.6) now provides an interrelation between these two generating functions:

$$
\mathfrak{G}_q^{(\lambda)}(\cos\theta, t) = \exp\left(\frac{1 - q^{\lambda\mathcal{D}}}{1 - q^{\mathcal{D}}} \ln \mathfrak{G}^{(1)}(\cos\theta, t)\right), \tag{4.7}
$$

where $\mathcal{D} \equiv t\partial_t$. For the purpose of simplicity we, in the above expression, have used the classical generating function for the $\lambda = 1$ case, namely $\mathfrak{G}^{(1)}(\cos\theta, t)$. In this case the Gegenbauer polynomials reduce to the Chebyshev polynomials of the second kind.

Starting from the mapping (4.7) of the deformed generating function on the classical generating function it is possible to develop via the route adopted in sections 3 and 2 a general nonlinear connection formula between an arbitrary q-Gegenbauer polynomial and its classical partners. But since the general formula is notationally quite cumbersome, we subsequently express the first few q-Gegenbauer polynomials in terms of their classical analogues. But prior to that it is worthwhile to recast (4.7) in another form particularly suitable for deriving a set of *sum rules*:

$$\ln \mathfrak{G}_q^{(\lambda)}(\cos\theta, t) = [\lambda]_{q^{\mathcal{D}}} \ln \mathfrak{G}^{(1)}(\cos\theta, t). \tag{4.8}$$

Expanding both sides of (4.8) in the variable t and equating its identical powers, we obtain the general sum rule:

$$\sum_{n\in\mathbb{N}} \frac{(-1)^{n+1}}{n} \sum_{\substack{\ell_1,\dots,\ell_n \\ \in\mathbb{N}}} C_{\ell_1}^{(\lambda)}(z;q) \cdots C_{\ell_n}^{(\lambda)}(z;q)\delta_{\ell_1+\cdots+\ell_n,\mathcal{N}}$$

$$= [\lambda]_{q^{\mathcal{N}}} \sum_{n\in\mathbb{N}} \frac{(-1)^{n+1}}{n} \sum_{\substack{\ell_1,\dots,\ell_n \\ \in\mathbb{N}}} C_{\ell_1}(z) \cdots C_{\ell_n}(z)\delta_{\ell_1+\cdots+\ell_n,\mathcal{N}}. \tag{4.9}$$

As mentioned in the context of (4.7), we here and henceforth consider the classical Gegenbauer polynomials $C_n^{(\lambda)}(z)$ for the $\lambda = 1$ case and suppress the superscript. For an explicit value of $\mathcal{N} \in \mathbb{N}$ the sum rule immediately follows from (4.9). To illustrate the first few cases we introduce, via (4.9), a set of variables:

$$\mathcal{I}_1^{(\lambda)}(z;q) = C_1^{(\lambda)}(z;q) \qquad \mathcal{I}_2^{(\lambda)}(z;q) = C_2^{(\lambda)}(z;q) - \tfrac{1}{2}\big(C_1^{(\lambda)}(z;q)\big)^2$$

$$\mathcal{I}_3^{(\lambda)}(z;q) = C_3^{(\lambda)}(z;q) - C_1^{(\lambda)}(z;q)C_2^{(\lambda)}(z;q) - \tfrac{1}{3}\big(C_1^{(\lambda)}(z;q)\big)^3$$

$$\mathcal{I}_4^{(\lambda)}(z;q) = C_4^{(\lambda)}(z;q) - C_1^{(\lambda)}(z;q)C_3^{(\lambda)}(z;q) - \tfrac{1}{2}\big(C_2^{(\lambda)}(z;q)\big)^2$$

$$\qquad + \big(C_1^{(\lambda)}(z;q)\big)^2 C_2^{(\lambda)}(z;q) - \tfrac{1}{4}\big(C_1^{(\lambda)}(z;q)\big)^4$$

$$\mathcal{I}_5^{(\lambda)}(z;q) = C_5^{(\lambda)}(z;q) - C_1^{(\lambda)}(z;q)C_4^{(\lambda)}(z;q) - C_2^{(\lambda)}(z;q)C_3^{(\lambda)}(z;q) + \big(C_1^{(\lambda)}(z;q)\big)^2 C_3^{(\lambda)}(z;q)$$

$$\qquad + C_1^{(\lambda)}(z;q)\big(C_2^{(\lambda)}(z;q)\big)^2 - \big(C_1^{(\lambda)}(z;q)\big)^3 C_2^{(\lambda)}(z;q) + \tfrac{1}{5}\big(C_1^{(\lambda)}(z;q)\big)^5. \tag{4.10}$$

The sum rules given in (4.9) may then be succinctly stated as

$$\mathcal{I}_\ell^{(\lambda)}(z;q) = [\lambda]_{q^\ell}\big(\mathcal{I}_\ell^{(\lambda=1)}(z;q)\big)_{q\to 1} \qquad \forall \ell \in \mathbb{N}. \tag{4.11}$$

We also enumerate a few explicit examples of our nonlinear connection formula relating q-Gegenbauer polynomials and their classical counterparts:

$$C_0^{(\lambda)}(z;q) = C_0(z) = 1, \qquad C_1^{(\lambda)}(z;q) = [\lambda]_q C_1(z),$$

$$C_2^{(\lambda)}(z;q) = [\lambda]_{q^2} C_2(z) - \tfrac{1}{2}\big([\lambda]_{q^2} - [\lambda]_q^2\big)C_1^2(z),$$

$$C_3^{(\lambda)}(z;q) = [\lambda]_{q^3} C_3(z) - ([\lambda]_{q^3} - [\lambda]_q[\lambda]_{q^2})C_1(z)C_2(z)$$

$$\qquad + \tfrac{1}{6}\big(2[\lambda]_{q^3} - 3[\lambda]_q[\lambda]_{q^2} + [\lambda]_q^3\big)C_1^3(z),$$

$$C_4^{(\lambda)}(z;q) = [\lambda]_{q^4} C_4(z) - \tfrac{1}{2}\big([\lambda]_{q^4} - [\lambda]_{q^2}^2\big)C_2^2(z) - ([\lambda]_{q^4} - [\lambda]_q[\lambda]_{q^3})C_1(z)C_3(z)$$

$$\qquad + \tfrac{1}{2}\big(2[\lambda]_{q^4} - 2[\lambda]_q[\lambda]_{q^3} - [\lambda]_{q^2}^2 + [\lambda]_q^2[\lambda]_{q^2}\big)C_1^2(z)C_2(z)$$

$$\qquad - \tfrac{1}{24}\big(6[\lambda]_{q^4} - 8[\lambda]_q[\lambda]_{q^3} - 3[\lambda]_{q^2}^2 + 6[\lambda]_q^2[\lambda]_{q^2} - [\lambda]_q^4\big)C_1^4(z),$$

$$\begin{aligned}
C_5^{(\lambda)}(z;q) = {}& [\lambda]_{q^5} C_5(z) - ([\lambda]_{q^5} - [\lambda]_q [\lambda]_{q^4}) C_1(z) C_4(z) - ([\lambda]_{q^5} - [\lambda]_{q^2}[\lambda]_{q^3}) C_2(z) C_3(z) \\
& + \tfrac{1}{2}\big(2[\lambda]_{q^5} - 2[\lambda]_q[\lambda]_{q^4} - [\lambda]_{q^2}[\lambda]_{q^3} + [\lambda]_q^2[\lambda]_{q^3}\big) C_1^2(z) C_3(z) \\
& + \tfrac{1}{2}\big(2[\lambda]_{q^5} - [\lambda]_q[\lambda]_{q^4} - 2[\lambda]_{q^2}[\lambda]_{q^3} + [\lambda]_q[\lambda]_{q^2}^2\big) C_1(z) C_2^2(z) \\
& - \tfrac{1}{6}\big(6[\lambda]_{q^5} - 6[\lambda]_q[\lambda]_{q^4} - 5[\lambda]_{q^2}[\lambda]_{q^3} + 3[\lambda]_q^2[\lambda]_{q^3} \\
& + 3[\lambda]_q[\lambda]_{q^2}^2 - [\lambda]_q^3[\lambda]_{q^2}\big) C_1^3(z) C_2(z) + \tfrac{1}{120}\big(24[\lambda]_{q^5} - 30[\lambda]_q[\lambda]_{q^4} \\
& - 20[\lambda]_{q^2}[\lambda]_{q^3} + 20[\lambda]_q^2[\lambda]_{q^3} + 15[\lambda]_q[\lambda]_{q^2}^2 - 10[\lambda]_q^3[\lambda]_{q^2} + [\lambda]_q^5\big) C_1^5(z). \quad (4.12)
\end{aligned}$$

Explicit use of the above connection formulae immediately proves the validity of the sum rules presented in (4.11). It is interesting to note that modulo a multiplicative factor, the combinations $\mathcal{I}_\ell^{(\lambda)}(z;q)$ preserve their 'form' when the deformed polynomials are recast in terms of their classical partners via our connection formulae.

5. Discussions

Using a realization of the *q*-exponential function as an infinite multiplicative series of the ordinary exponentials we have obtained a new set of nonlinear connection formulae for the *q*-orthogonal polynomials in terms of their classical partners. The scheme has been illustrated for *q*-Hermite, *q*-Laguerre and *q*-Gegenbauer polynomials. One significance of our result is that these connection formulae may be useful in understanding the spectrum and the matrix elements of the *q*-deformed systems in terms of their classical analogues and explicitly calculable perturbations thereof. This is expected to provide better physical comprehension of the models such as *q*-deformed rotator [17, 18], which found wide applications in nuclear physics.

We conclude our work by mentioning some other possible applications of our procedure. The connection formulae described here may also be developed for the little *q*-Jacobi polynomials. The matrix elements of the finite-dimensional unitary co-representations of the quantum group $SU_q(2)$ and the supergroup $OSp_q(1/2)$ may be obtained [16, 19] using the little *q*-Jacobi polynomials. Our formulation allows expressing the said matrix elements via their classical analogues. It may, for instance, be useful in constructing so far unknown generating function [20] of the co-representation matrices of the quantum supergroup $OSp_q(1/2)$. We will address this issue elsewhere.

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